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Construction of Product-systems

PhD Theses

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Introduction

My work is about generalization of Haar-like wavelets. Our constructions are based on the special properties of three classical systems: Rademacher-, Walsh-Paley- and Haar-systems. We used a simple and general method to construct orthogonal or biorthogonal systems starting from the Dirichlet-kernel of the systems. Discrete orthogonal polynomial systems can be constructing by the Christoffel-Darboux-formula. Paley has proved that the 2^nth Dirichlet-kernel of Walsh-Paley-system can be written in a simple form. This property based on the fact, the product system of Rademacher system is the Walsh system. The same formula is the key at the construction of Haar-like systems. The Fourier-coefficients with respect to UDMD product system can be computed by an FFT-like fast algorithm and a similar methode can be used for the reconstruction.

At the end of the work we introduce new rational interpolation operators for the upper and lower half plane using the Malmquist-Takenaka systems of these Hardy spaces. Combining this two interpolations we can give exact interpolation for a large class of rational functions.

Thesis 1. Walsh-Chebyshev systems

Let us start our construction from the twofold map $A(x) := 2x^2 - 1$ $(x \in X)$ os set $X = [-1, 1]$ and our basic function be the identic map $\phi(x) := x$ $(x \in X)$.

The $T_{2^n}(x)$ $(x \in X, n \in \mathbb{N})$ subsystem of Chebyshev polynomials can be derivate from the twofold map $A(x) = T_2(x) = \cos(2 \arccos x)$ by iteration.

We can construct another product system Ψ_m $(m \in \mathbb{N})$. The product systems ψ_m $(m \in \mathbb{N})$ and Ψ_m $(m \in \mathbb{N})$ are biorthogonal with respect to the discrete scalar product of the set X^n.

Theorem 1. *[11] Let* $\Phi_k = 2\phi_k/(\phi_{k+1} + 1) \quad (k \in \mathbb{N})$ *and denote*

$$\Psi_m = \prod_{k=0}^{n-1} \Phi_k^{m_k} \ (0 \le m < 2^n)$$

the product system of Φ_k $(k = 0, 1, \cdots, n-1)$*. Then the systems* $(\psi_m, 0 \le m < 2^n)$ *and* $(\Psi_m, 0 \le m < 2^n)$ *are biorthogonal, ie* $\langle \psi_k, \Psi_\ell \rangle = \delta_{k\ell} \quad (0 \le k, \ell < 2^n)$.

Theorem 2. *[11]The mixed Dirichlet kernel of the product systems satisfies* $D_{2^n}(s,t) = 2^n \delta_{st}$, $(s, t \in X^n)$*. Moreover the polynomial* Sf *interpolate the function* f *in the points of, i.e.* $(Sf)(x) = f(x)$ $(x \in \ X^n)$.

Thesis 2. Walsh-like systems

Denote $\mathcal{I}$ an interval in $\mathbb{R}$ and $A : \mathcal{I} \to \mathcal{I}$ be a twofold map on $\mathcal{I}$.

The function $\varphi^1, [\varphi^0]$ is called odd [even] with respect to the map A, if $A(x') = A(x'') = y$ and $x' \neq x''$ imply

$$\varphi^{(j)}(x') = (-1)^j \varphi^{(j)}(x'') \quad (j = 0,\ 1). \tag{2.1}$$

Let $A_0(x) := x$ and $A_{n+1}(x) := A(A_n(x))$ for every $x \in \mathcal{I}$. It is easy to see that $A_n : \mathcal{I} \to \mathcal{I}$ is a 2^n-fold map.

The functions $\varphi_n^{(1)}(x) := \varphi^{(1)}(A_n(x))$ $(n \in \mathbb{N})$ are called Rademacher-like functions.

The twofold map A is the analogue of the function $2x$ $(mod 1)$ in the original Rademacher-construction and the function $\varphi^{(1)}$ is the analogue of the basic function r.

Let us start from the following finite collection of the system $\phi_n = \{\varphi_n^{(0)},\ \varphi_n^{(1)}\}$, $(0 \le n < N)$.

Then

$$\Phi_\ell := \prod_{k=0}^{N-1} \varphi_k^{(\ell_k)}, \quad \ell = \sum_{k=0}^{N-1} \ell_k \cdot 2^k, \quad \ell_k \in \{0,\ 1\}$$

is the generalized product system of the system $\{\phi_n,\ 0 \le n < N\}$.

Let denote by $\{\Phi_\ell,\ 0 \le \ell < 2^N\}$ the product system of $\{\phi_n,\ 0 \le n < N\}$ and $\{\Psi_\ell,\ 0 \le \ell < 2^N\}$ the product system of $\{\Upsilon_n = \{\psi_n^{(0)},\ \psi_n^{(1)}\},\ 0 \le n < N\}$. Then the mixed Dirichlet kernel of the product systems is defined by

$$D_{2^N}(s,t) := \sum_{\ell=0}^{2^N-1} \Phi_\ell(s)\overline{\Psi}_\ell(t). \tag{2.2}$$

From the definition of the product system we get the following explicit form for the mixed Dirichlet kernel:

$$D_{2^N}(s,t) = \sum_{\ell=0}^{2^N-1} \Phi_\ell(s)\overline{\Psi}_\ell(t) = \sum_{\ell=0}^{2^N-1}\prod_{k=0}^{N-1} \varphi_k^{(\ell_k)}(s)\cdot\overline{\psi}_k^{(\ell_k)}(t) = \prod_{k=0}^{N-1}\left(\varphi_k^{(0)}(s)\cdot\overline{\psi}_k^{(0)}(t) + \varphi_k^{(1)}(s)\cdot\overline{\psi}_k^{(1)}(t)\right).$$

Denote Y_N the set of preimages of the fixed $x_0 \in \mathcal{I}$ on the map A_N. The set Y_N has exactly 2^N different elements:

$$Y_N := \{x \in \mathcal{I} : A_N(x) = x_0\} = \{x_k^N : k = 0, 1, \cdots, 2^N - 1\}. \tag{2.3}$$

Restricting the functions to the set Y_N we can discretize the generalized product system.

Let denote by $\{\Phi_\ell,\ 0 \le \ell < 2^N\}$ the generalized product system of the system $\{\phi_n = \{\varphi_n^{(0)},\ \varphi_n^{(1)}\},\ 0 \le n < N\}$ defined above.

Assume that $\varphi^{(0)}(x) \neq 0$ and $\varphi^{(1)}(x) \neq 0$, if $x \in Y_N$ and restart the construction from the finite set of functions $\Upsilon := \{\psi^{(0)},\ \psi^{(1)}\} = \left\{\frac{1}{\overline{\varphi}^{(0)}},\ \frac{1}{\overline{\varphi}^{(1)}}\right\}$.

Let denote by $\{\Psi_\ell,\ 0 \le \ell < 2^N\}$ the generalized product system of system $\Upsilon_n(x) = \{\psi_n^{(0)}(x),\ \psi_n^{(1)}(x)\} = \{\psi^{(0)}(A_n(x)),\ \psi^{(1)}(A_n(x))\}$, $(0 \le n < N)$.

Theorem 3. *[12]The generalized product system $\{\Phi_\ell,\ 0 \le \ell < 2^N\}$ and the generalized product system $\{\Psi_\ell,\ 0 \le \ell < 2^N\}$ defined above are biorthogonal with respect the discrete scalar product of the set Y_N.*

Theorem 4 (Interpolation formula)**.** *[12] Let define by $S_N f := \sum_{m=0}^{2^N-1}[f, \Psi_m]_{Y_N}\psi_m$ Then $S_N f$ is an interpolation formula for function f in the points of Y_N ie. $(S_N f)(x) = f(x)$, $x \in Y_N$.*

Thesis 3. Generalized Walsh-like Systems in 2D

Let us fixed the set $X \neq \emptyset$ and the functions $f_n^{(j)} : X \to \mathbb{C}$ $(0 \le j < m, 0 \le n < N)$, where $N, m \in \mathbb{N}^* := \{1, 2, \cdots\}$ and $m \ge 2$. We will consider the following finite collection of ordered sets of functions $\mathcal{F}_n = \{f_n^{(0)},\ f_n^{(1)}, \ldots,\ f_n^{(m-1)}\}$, $0 \le n < N$.

Definition 5. The system $\mathscr{F} := \{F_\ell \,|\, 0 \le \ell < m^N\}$ defined by

$$F_\ell := \prod_{k=0}^{N-1} f_k^{(\ell_k)}, \quad \ell = \sum_{k=0}^{N-1} \ell_k \cdot m^k,\ \ell_k \in \{0, 1, \ldots, m-1\} \tag{3.1}$$

is called the **generalized product system** generated by the collection $(\mathcal{F}_n,\ n < N)$.

Obviously the product of the ordered sets of functions $\mathcal{F}_j$ $(0 \le j < N)$ is equal to $\mathscr{F}$: $\mathcal{F}_0 \cdot \mathcal{F}_1 \cdots \mathcal{F}_{N-1} = \{F_\ell|\ 0 \le \ell < m^N\} = \mathscr{F}$.

Let us denote by $\mathscr{G} := \{G_\ell \,|\, 0 \le \ell < m^N\}$ the generalized product system of the collection $\mathcal{G}_n = \{g_n^{(0)},\ g_n^{(1)}, \ldots, g_n^{(m-1)}\}$, $0 \le n < N$.

Then the **mixed Dirichlet kernels of the systems** $\mathscr{F}$ and $\mathscr{G}$ are defined by $D_n(s,t) := \sum_{\ell=0}^{n-1} F_\ell(s)\overline{G}_\ell(t)$ $(n \in \mathbb{N})$.

Lemma 6. *In the case $n = m^N$ the mixed Dirichlet kernel of the product systems can be written in a product form*

$$D_{m^N}(s,t) = \sum_{\ell=0}^{m^N-1} \prod_{k=0}^{N-1} f_k^{(\ell_k)}(s) \cdot \overline{g}_k^{(\ell_k)}(t) = \prod_{i=0}^{N-1} \sum_{j=0}^{m-1} f_i^{(j)}(s) \cdot \overline{g}_i^{(j)}(t). \tag{3.2}$$

Definition 7. Let X be a nonempty set and $m \in \mathbb{N}^*$ a fixed number. The function $A : X \to X$ is called **m-fold map of** X, if for every $y \in X$ there exist exactly m different points $x_0, x_1, \cdots, x_{m-1} \in X$, such that $A(x_0) = A(x_1) = \cdots = A(x_{m-1}) = y$.

Definition 8. The systems $\mathcal{F} := \{f^{(0)}, f^{(1)}, \cdots, f^{(m-1)}\}$ and $\mathcal{G} := \{g^{(0)}, g^{(1)}, \cdots, g^{(m-1)}\}$ are called **biorthogonal with respect to the m-fold map** A of X, if for every $y \in X$

$$\frac{1}{m} \sum_{x \in A^{-1}(y)} f^{(i)}(x)\overline{g}^{(j)}(x) = \delta_{ij} \quad (0 \le i, j < m). \tag{3.3}$$

In the case $\mathcal{F} = \mathcal{G}$ we say that the system $\mathcal{F}$ is **orthogonal with respect to the m-fold map** A.

Corollary 9. *If the function $f^{(0)}$ is even and $f^{(1)}$ is odd with respect to the 2-fold map A and $|f^{(0)}(x)| = |f^{(1)}(x)| = 1$, then the system $\mathcal{F} = \{f^{(0)}, f^{(1)}\}$ is orthogonal with respect to the map A.*

In our construction we need the following iteration of the m-fold map A. Let $A_0(x) := x$ and $A_{n+1}(x) := (A \circ A_n)(x) := A(A_n(x))$ for every $x \in X$. It is clear that A_n is an m^n-fold map of X.

To discretize the generalized product system we introduce the set $X_N := \{x \in X : A_N(x) = x_0\} = \{x_k^N \,|\, k = 0, 1, \cdots, m^N - 1\}$ and the scalar product on X_N: $[f,g]_{X_N} := \frac{1}{m^N} \sum_{x \in X_N} f(x)\overline{g}(x)$.

To get biorthogonal product systems we start from the sets of functions $\mathcal{F}$ and $\mathcal{G}$ and introduce the collections of functions $\mathcal{F}_n := \{f \circ A_n \,|\, f \in \mathcal{F}\}$ and $\mathcal{G}_n = \{g \circ A_n \,|\, g \in \mathcal{G}\}$.

Denote by $\mathscr{F} := \{F_\ell,\ 0 \le \ell < m^N\}$ and $\mathscr{G} := \{G_\ell,\ 0 \le \ell < m^N\}$ the generalized product systems generated by $\{\mathcal{F}_n,\ 0 \le n < N\}$ and $\{\mathcal{G}_n,\ 0 \le n < N\}$.

Theorem 10. *[13] Suppose that the systems $\mathcal{F}$ and $\mathcal{G}$ are biorthogonal with respect to m-fold map A of X. Then the product systems $\mathscr{F}$ and $\mathscr{G}$ generated by $\mathcal{F}$ and $\mathcal{G}$ and by A are biorthogonal with respect to the discrete scalar product of the set X_N, i.e. $[F_\ell, G_k]_{X_N} = \delta_{\ell k}$, $(0 \le \ell,\ k < m^N)$. Moreover the mixed Dirichlet kernel of the generalized product systems satisfies $D_{m^N}(s,t) = m^N \delta_{st}$, $(s, t \in X_N)$.*

Denote

$$S_n f := \sum_{k=0}^{n-1} [f, G_k]_{X_N} F_k \quad (0 \le n \le m^N)$$

the partial sums of the biorthogonal expansion of $f : X \to \mathbb{C}$ with respect the systems $\mathscr{F}$ and $\mathscr{G}$.

Corollary 11. *The partial sum $S_{m^N} f$ interpolate the function f at the point of X_N ie. $(S_{m^N} f)(x) = f(x)$, $(x \in X_N)$.*

For functions $A_1, A_2 : X \to X$ we introduce the Kronecker product $X^2 := X \times X$ and the map $A : X^2 \to X^2$ defined by $A(x) := (A_1 \diamond A_2)(x) := (A_1(x_1), A_2(x_2))$, $(x = (x_1, x_2) \in X^2)$.

Theorem 12. *[13] Suppose that the systems $\mathcal{F}$ and $\mathcal{G}$ are biorthogonal with respect to m-fold map A of X. Then the Kronecker product $\mathcal{F}^2 := \mathcal{F} \times \mathcal{F}$ and $\mathcal{G}^2 := \mathcal{G} \times \mathcal{G}$ are biorthogonal with respect to the m^2-fold map $A^2 := A \diamond A$ of X^2. Thus the product systems generated $\mathcal{F}^2$ and $\mathcal{G}^2$ are biorthogonal.*

Thesis 4. Generalized Haar-Fourier Transform

Let us fix the set $X \neq \emptyset$ and let us denote a two-fold map by $A : X \to X$. We can define the iterated map of A by $A^0(x) = x$, and $A^n := A^{n-1} \circ A$ $(x \in X, n \in \mathbb{N}^*)$. Starting from a fixed element $x_0^0 \in X$ we can define the preimage of this element in map A^n, this discrete set is denoted by X_n. The elements of the set $X_n = \{x_k^n : 0 \leq k < 2^n\}$ can be indexed such that the following will be true $A^{-1}(\{x_k^n\}) = \{x_{2k+1}^{n+1}, x_{2k}^{n+1}\}$, and consequently $A(x_\ell^{n+1}) = x_{[\ell/2]}^n$, where $[s]$ denoted the integer part of the real number s. From this by induction we get $A^j(x_\ell^n) = x_{[\ell 2^{-j}]}^{n-j}$ $(0 \leq \ell < 2^n, 0 \leq j \leq n)$.

Let us fix $N \in \mathbb{N}^*$. Let us define the following subsets of the set X_N: $I_{n,k} := A^{n-N}(\{x_k^n\})$, where $0 \leq k < 2^n$, and $0 \leq n \leq N$. The set $I_{n,k}$ has 2^{N-n} elements and is called dyadic interval of the set X_N. These subsets have similar properties as intervals $[k2^{-n}, (k+1)2^{-n})$:

$$\begin{aligned} I_{0,0} = A^{-N}(\{x_0^0\}) = X_N, \ I_{N,k} = A^0(\{x_k^N\}) = \{x_k^N\}, \\ I_{n+1,2k+1} \cup I_{n+1,2k} = I_{n,k} \quad (0 \leq k < 2^n, 0 \leq n < N). \end{aligned} \tag{4.1}$$

Starting from the function $\varphi : X \to \mathbb{C}$ by using the iterated map of the two-fold map $A : X \to X$ we can define a sequence of functions: $(\varphi_n, n \in \mathbb{N})$, where $\varphi_n(x) := \varphi(A^n(x))$, $(x \in X, n \in N)$. The system $(\varphi_n, n \in \mathbb{N})$ is called Rademacher-like system.

The systems φ^j, $\psi^j : X \to \mathbb{C}$ $(j = 0, 1)$ are called biorthogonal with respect to the two-fold map A, if for $A(x') = A(x'') = x, x' \neq x''$ and for $i, j = 0, 1$ the following is satisfied

$$\frac{1}{2} \sum_{z \in A^{-1}(x)} \varphi^i(z)\overline{\psi}^j(z) = \frac{1}{2}(\varphi^i(x')\overline{\psi}^j(x') + \varphi^i(x'')\overline{\psi}^j(x'')) = \delta_{ij}. \tag{4.2}$$

The product system of a Rademacher-like system is called Walsh-like system.

The mixed Dirichlet-kernel of these systems can be written in the product form:

$$D_{2^N}(x, y) := \sum_{m=0}^{2^N-1} \Phi_m(x)\overline{\Psi}_m(y) = \prod_{j=0}^{N-1} (\varphi_j^0(x)\overline{\psi}_j^0(y) + \varphi_j^1(x)\overline{\psi}_j^1(y)). \tag{4.3}$$

Using the function $L(x, y) = \varphi^0(x)\overline{\psi}^0(y) + \varphi^1(x)\overline{\psi}^1(y)$, $(x, y \in X)$ the Dirichlet-kernel D_{2^N} can be written as

$$D_{2^N}(x, y) := \prod_{j=0}^{N-1} L(A^{N-1-j}(x), A^{N-1-j}(y)) \quad (x, y \in X).$$

Let us introduce the following analogues of the scaling functions $\chi_{n,k}$:

$$\mathcal{I}_{n,k}(x) := 2^{-n} \prod_{j=0}^{n-1} L(A^{N-1-j}(x), A^{N-1-j}(x_k^n)), \quad (x \in X, 0 \leq k < 2^n, n = 1, 2, \cdots, N). \tag{4.4}$$

Theorem 13. *[14] If the biorthogonal relation (4.2) is satisfied then for the scaling functions $\mathcal{I}_{n,k}$ on the set X_N we have*

$$\mathcal{I}_{n,k} = \chi_{I_{n,k}} \quad (0 \leq k < 2^n, \ n = 1, 2, \cdots, N), \tag{4.5}$$

thus in the points of the set X_N the following scaling equation is true

$$\mathcal{I}_{n+1,2k}(x) + \mathcal{I}_{n+1,2k+1}(x) = \mathcal{I}_{n,k}(x), \quad (x \in X_N, 0 \leq k < 2^n, \ n = 1, 2, \cdots, N). \tag{4.6}$$

Similarly to the original properties we can introduce the Haar-like functions:

$$\mathcal{H}_{n,k} := \mathcal{I}_{n+1,2k} - \mathcal{I}_{n+1,2k+1} \quad (0 \leq k < 2^n, n = 0, 1, \cdots, N-1).$$

Theorem 14. *[14]The system $\mathcal{H}_{n,k}$ $(0 \le k < 2^n, n = 0, 1, \cdots, N-1)$ is a discrete orthogonal Haar-like system with respect to the scalar product of the set X_N, ie.*

$$\langle \mathcal{H}_{n,k}, \mathcal{H}_{m,\ell} \rangle = 2^{-n} \cdot \delta_{n,m} \cdot \delta_{k,\ell}, \quad (0 \le k < 2^n,\ 0 \le n < N,\ 0 \le \ell < 2^m,\ 0 \le m < N).$$

Thesis 5. Discrete Orthogonal and Biorthogonal Product Systems

Let us fix the number $p \in \mathbb{N}^{**} := \{2, 3, \cdots\}$ and the non-empty set X. Let us start from the following finite collection of the systems $\phi_n = (\varphi_n^{(i)}, 0 \le i < p)$, where $0 \le n < N \le \infty$ and $\varphi_n^{(i)} : X \to \mathbb{C}$.

Definition 15. The **generalized product system** of the systems $\{\phi_n,\ 0 \le n < N\}$ is generated by the following construction.

$$\Phi_m = \prod_{k=0}^{N-1} \varphi_k^{(m_k)}, \quad m = \sum_{k=0}^{N-1} m_k \cdot p^k \quad m_k \in \{0, 1, \ldots, p-1\}. \tag{5.1}$$

Definition 16. The map $A : X \to X'$ is called **p-fold map on set** X if every $x \in X'$ has exactly p preimages.

Definition 17. Let $f^{(j)}, g^{(j)} : X \to \mathbb{C}, \quad (j = 0, 1, \ldots, p-1)$, two finite collection of functions and let $\rho : X \to (0, +\infty)$ be a positive weight-function. The system $F = (f^{(0)}, f^{(1)}, \ldots f^{(p-1)})$ and the system $G = (g^{(0)}, g^{(1)}, \ldots g^{(p-1)})$ is called (A, ρ)**-biorthogonal** if for every $x \in X'$

$$\sum_{t \in A^{-1}(x)} f^{(i)}(t) \cdot \overline{g^{(j)}(t)} \rho(t) = \sum_{k=0}^{p-1} f^{(i)}(x_k) \cdot \overline{g^{(j)}(x_k)} \rho(x_k) = \delta_{ij}, \tag{5.2}$$

if $F = G$ the system F is called (A, ρ)**-orthonormal**.

Let us consider the elements $A_n : X \to X, n \in \mathbb{N}^*$ of the sequence of p-fold maps, then for every $x \in X$ pontra the set $A_n^{-1}(x) \subset X$ contains exactly p elements for every n.

The composition of these maps will generate a sequence of maps

$$T_0(x) := x\ (x \in X),\ T_n = A_n \circ A_{n-1} \circ \ldots \circ A_1 = A_n \circ T_{n-1}\ (n \in \mathbb{N}^*). \tag{5.3}$$

It is easy to see, that map T_n generated in this way is a p^n-fold map on set X. Starting from a fixed element $x_0 \in X$ let us define the preimages of this element in map T_n. This discrete set is denoted by X_n.

Definition 18. Let us fix $N \in \mathbb{N}^*$. Let us define the $I_{n,k} := A^{n-N}(\{x_k^n\})$ $(0 \le k < p^n, 0 \le n \le N)$ subsets of the set X_N. The set $I_{n,k}$ has p^{N-n} elements and is called **p-adic interval of the set X_N.**

Denote $F_n := (f_n^{(0)}, f_n^{(1)}, \ldots, f_n^{(p-1)})$ and $G_n := (g_n^{(0)}, g_n^{(1)}, \ldots, g_n^{(p-1)})$ two sequences of systems and $\rho_n : X \to (0, +\infty)$ $(n \in \mathbb{N})$ a sequence of positive weight functions, where $f_n^{(i)}, g_n^{(i)} : X \to \mathbb{C}$ $(i = 0, 1, \ldots, p-1)$ $(n \in \mathbb{N})$. Assume that system F_n and system G_n are (A_{n+1}, ρ_n) biorthogonals for every index $0 \le n < N$.

By composition construct the systems $\phi_n^{(i)} := f_n^{(i)} \circ T_n$, and $\gamma_n^{(i)} := g_n^{(i)} \circ T_n$ for every $0 \le i < p$ and $n \in \mathbb{N}$ and from them the systems $\Phi_n := (\phi_n^{(i)}, 0 \le i < p)$ and $\Gamma_n := (\gamma_n^{(i)}, 0 \le i < p)$.

The systems were constructed the previous way are called Rademacher-like functions and the product systems of these systems are called Walsh-like systems and can be written in form

$$\psi_m := \prod_{i=0}^{N-1} \phi_i^{m_i} \quad \eta_m := \prod_{i=0}^{N-1} \gamma_i^{m_i} \quad (0 \le m < p^N) \quad m = \sum_{i=0}^{N-1} m_i p^i.$$

Theorem 19 ([16]). *The p^n-th mixed kernel of the system $(\psi_m, 0 \le m < p^N)$ and system $(\eta_m, 0 \le m < p^N)$ can be written in product form*

$$D_{p^n}(x,t) := \sum_{k=0}^{p^n-1} \psi_k(x)\overline{\eta_k(t)} = \prod_{i=0}^{n-1}(\phi_i^{(0)}(x)\overline{\gamma}_i^{(0)}(t) + \phi_i^{(1)}(x)\overline{\gamma}_i^{(1)}(t) + \ldots + \phi_i^{(p-1)}(x)\overline{\gamma}_i^{(p-1)}(t)),$$

where $x, t \in X$ and $0 \le n < N$.

Theorem 20 ([16]). *The product systems $\psi_m := \prod_{i=0}^{N-1} \phi_i^{m_i}$ and $\eta_m := \prod_{i=0}^{N-1} \gamma_i^{m_i}$ where $0 \le m < p^N$ and $m = \sum_{i=0}^{N-1} m_i p^i$, are biorthogonal with respect to the discrete scalarproduct*

$$\langle f, g\rangle := \sum_{x \in X_N} f(x)\overline{g(x)}\sigma_N(x),$$

where $\sigma_N(x) D_{p^N}(x,t) = \delta_{x,t}$ and $\sigma_N(x) := \prod_{i=0}^{N-1} \rho_i(T_i(x))$ $(x \in X_N)$.

Corollary 21. *For any function f the p^N-th partial sum of Fourier series is*

$$(S_{p^N}f)(x) = \sum_{k=0}^{p^N-1} \langle f, \eta_k\rangle \psi_k(x) = \sum_{t \in X_N} f(t) \sum_{k=0}^{p^N-1} \psi_k(x)\overline{\eta_k(t)}\sigma_N(t) = \sum_{t \in X_N} f(t) D_{p^N}(x,t)\sigma_N(t).$$

From this $(S_{p^N}f)(x) = f(x)$ $x \in X_N$.

The p^{n+1}-th Dirichlet kernel can be written in the product form

$$D_{p^n}(x,t) := \sum_{k=0}^{p^n-1} \psi_k(x)\overline{\eta_k(t)} = \prod_{i=0}^{n-1}(\phi_i^{(0)}(x)\overline{\gamma}_i^{(0)}(t) + \phi_i^{(1)}(x)\overline{\gamma}_i^{(1)}(t) + \ldots + \phi_i^{(p-1)}(x)\overline{\gamma}_i^{(p-1)}(t)).$$

Using the function

$$L(x,y) = \phi^{(0)}(x)\overline{\gamma^{(0)}(y)} + \phi^{(1)}(x)\overline{\gamma^{(1)}(y)} + \ldots + \phi^{(p-1)}(x)\overline{\gamma^{(p-1)}(y)} \quad (x, y \in X)$$

the Dirichlet-kernel D_{p^N} can be written as

$$D_{p^N}(x,y) := \prod_{j=0}^{N-1} L(A^{N-1-j}(x), A^{N-1-j}(y)) \quad (x, y \in X).$$

Let us introduce the following analogues of the scaling functions $\chi_{n,k}$:

$$\mathcal{I}_{n,k}(x) := p^{-n} \prod_{j=0}^{n-1} L(A^{N-1-j}(x), A^{N-1-j}(x_k^n)), \quad (x \in X, 0 \le k < p^n, n = 1, 2, \cdots, N).$$

Theorem 22 ([16]). *The scaling functions $\mathcal{I}_{n,k}$, which were made from orthonormal functions, satisfies on the set X_N*

$$\mathcal{I}_{n,k} = \chi_{I_{n,k}} \quad (0 \le k < p^n, n = 1, 2, \ldots, N),$$

thus in the points of the set X_N the following scaling equation is true

$$\mathcal{I}_{n+1,pk}(x) + \mathcal{I}_{n+1,pk+1}(x) + \ldots + \mathcal{I}_{n+1,pk+p-1}(x) = \mathcal{I}_{n,k}(x), \quad (x \in X_N, 0 \le k < p^n, n = 1, 2, \cdots, N).$$

We can introduce the Haar-like functions by the following equations

$$\mathcal{H}_{n,k}^{(j)} := \mathcal{I}_{n,k}\varphi_{n,k}^{(j)}, \text{ és } \mathcal{K}_{n,k}^{(j)} := \mathcal{I}_{n,k}\gamma_{n,k}^{(j)} \quad (0 \le k < p^n, n = 0, 1, \cdots, N-1,\ j = 1, \dots, p-1).$$

Theorem 23 ([16]). *The discretized of system $\mathcal{H}_{n,k}^{(j)}$ and the discretized of system $\mathcal{K}_{n,k}^{(j)}$ is a discrete biorthogonal system with respect to the scalar product $\langle f, g\rangle := p^{-N}\sum_{x\in X_N} f(x)\overline{g}(x)$, ie.*

$$\langle \mathcal{H}_{n,k}^{(j)}, \mathcal{K}_{m,\ell}^{(i)}\rangle = p^{-n}\cdot\delta_{n,m}\cdot\delta_{k,\ell}\cdot\delta_{i,j}, \quad (0 \le k < p^n,\ 0 \le n < N,\ 0 \le \ell < p^m,\ 0 \le m < N,\ 1 \le i, j < p).$$

Thesis 6. Rational Interpolation and Applications

The construction of the following orthogonal and discrete orthogonal systems is different to previous constructions but efficient interpolation formula can be deriveted by these systems too. Our results were published in [15].

A generalization of the Fourier-type representation is analyzed using special rational orthogonal bases: the Malmquist-Takenaka system for the upper and lower half plane. Based on the discrete orthogonality of the Malmquist-Takenaka system we introduce new rational interpolation operators for the upper and lower half plane as well. Combining this two interpolations we can give exact interpolation for a large class of rational functions among them for Runge test function. We study the properties of these rational interpolation operators.

Set $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, $\mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$ and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathrm{H}(\mathbb{C}_+)$, $\mathrm{H}(\mathbb{C}_-)$, $\mathrm{H}(\mathbb{D})$ the set of holomorphic functions in $\mathbb{C}_+$, $\mathbb{C}_-$respectively $\mathbb{D}$.

We shall work on the Hardy spaces
$\mathrm{H}^2(\mathbb{C}_+) = \left\{h \in \mathrm{H}(\mathbb{C}_+) : sup\left\{\int_{\mathbb{R}} |h(x+iy)|^2\, dx : y > 0\right\} < \infty\right\}$,
$\mathrm{H}^2(\mathbb{C}_-) = \left\{h \in \mathrm{H}(\mathbb{C}_-) : sup\left\{\int_{\mathbb{R}} |h(x+iy)|^2\, dx : y < 0\right\} < \infty\right\}$,
$\mathrm{H}^2(\mathbb{D}) = \left\{h \in \mathrm{H}(\mathbb{D}) : sup\left\{\int_{-\infty}^{\infty} |h(re^{it})|^2\, dt : r \in (0,1)\right\} < \infty\right\}$.

$\mathrm{H}^2(\mathbb{C}_+)$, $\mathrm{H}^2(\mathbb{C}_-)$ és $\mathrm{H}^2(\mathbb{D})$ are Hilbert spaces endowed with the following inner products
$\langle f, g\rangle_{\mathrm{H}^2(\mathbb{C}_+)} := \int_{\mathbb{R}} f(t)\overline{g(t)}dt,\ f, g \in \mathrm{H}^2(\mathbb{C}_+)$, $\langle f, g\rangle_{\mathrm{H}^2(\mathbb{C}_-)} := \int_{\mathbb{R}} f(t)\overline{g(t)}dt,\ f, g \in \mathrm{H}^2(\mathbb{C}_-)$,
and $\langle f, g\rangle_{\mathrm{H}^2(\mathbb{D})} := \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(e^{it})\overline{g(e^{it})}dt,\ f, g \in \mathrm{H}^2(\mathbb{D})$.

The Hardy space of the upper-half plane and the Hardy space of the unit disc can be connected through the Cayley transform which maps $\mathbb{C}_+$ to $\mathbb{D}$ and is defined by $K(\omega) = \frac{i-\omega}{i+\omega}$, $\omega \in \mathbb{C}_+$.

Let $\{\lambda_i\}_{i=0}^{\infty}$ an arbitrary sequence of complex numbers which lie in the upper half plane $\mathbb{C}_+$, and let $\{\Psi_n\}_{n=0}^{\infty}$ defined by

$$\Psi_1(z) = \frac{\sqrt{\frac{\Im\lambda_1}{\pi}}}{z - \overline{\lambda_1}}, \quad \Psi_n(z) = \frac{\sqrt{\frac{\Im\lambda_n}{\pi}}}{z - \overline{\lambda_n}}\prod_{k=1}^{n-1}\frac{z - \lambda_k}{z - \overline{\lambda_k}}, \quad (n = 2, 3, \dots). \tag{6.1}$$

This is a system of rational functions associated with the set of poles $\{\overline{\lambda_i}\}_{i=0}^{\infty}$ lying in the lower half-plane. The linear-fractional transformation $z = i\frac{1-y}{1+y}$ changes this system into the Malmquist-Takenaka system over the unit circle.

The system of functions $\{\Psi_n\}_{n=0}^{\infty}$ is orthonormal on the entire axis $-\infty < x < +\infty$ in the following sense $\int\limits_{-\infty}^{+\infty} \Psi_n(x)\overline{\Psi_m(x)}dx = \delta_{mn}$.

Moreover, if we have the following non Blaschke condition for the upper half plane

$$\sum_{k=1}^{\infty}\frac{\Im\lambda_k}{1+|\lambda_k|^2} = \infty, \tag{6.2}$$

then $\{\Psi_n\}_{n=1}^{\infty}$ is a complete orthonormal system for $\mathbb{H}^2(\mathbb{C}_+)$.

Let denote by $\mathcal{P}_k$ the space of polynomials of degree at most k, $\eta(z) = \prod_{n=1}^{N}(z-\overline{\lambda_n})$, $\omega(z) = \prod_{n=1}^{N}(z-\lambda_n)$ and set

$$\mathcal{R}_N := \left\{\frac{p}{\eta} : \ p \in \mathcal{P}_{N-1}\right\}, \quad \text{and} \quad \mathcal{R}_{\overline{N}} := \left\{\frac{p}{\omega} : \ p \in \mathcal{P}_{N-1}\right\}.$$

Accordingly one can set $\mathcal{R}_{N,\overline{N}} := \left\{\frac{p}{\eta\omega} : \ p \in \mathcal{P}_{2N-1}\right\}$. It is clear that $\mathcal{R}_{N,\overline{N}} = \mathcal{R}_N \oplus \mathcal{R}_{\overline{N}}$, i.e., they are orthogonal complement in $L^2(\mathbb{R})$.

The system $\mathbf{\Psi}_N = \{\Psi_n, \ n = 1,2,\ldots,N\}$ forms an orthonormal basis in $\mathcal{R}_N = \operatorname{span}\{\Psi_\ell, \ \ell = 1,\ldots,N\}$.

It is convenient to extend the system $\mathbf{\Psi}_N$ for negative indexes. Let denote by

$$\widetilde{B}_{\overline{N}}(z) = \prod_{k=1}^{N} \frac{z-\overline{\lambda_k}}{(z-\lambda_k)\eta_k},$$

and $\mathbf{\Psi}_{\overline{N}} = \{\Psi_{-n} = \widetilde{B}_{\overline{N}}\Psi_n, n = 1,2,\ldots,N\}$.

Since $\mathcal{R}_{\overline{N}} = \widetilde{B}_{\overline{N}}\mathcal{R}_N$, the system $\mathbf{\Psi}_{\overline{N}}$ forms an orthonormal basis in $\mathcal{R}_{\overline{N}}$, and $\mathcal{R}_{\overline{N}} = \operatorname{span}\{\Psi_\ell, \ \ell = -1,\ldots,-N\}$. $\mathcal{R}_{\overline{N}}$ is a subset of the Hardy space of the lower half plane.

Let us consider the orthogonal projection operator of an arbitrary function $f \in \mathrm{H}^2(\mathbb{C}_+)$ on the subspace $\mathcal{R}_N$ given by $P_N f(z) = \sum_{k=1}^{N}\langle f, \Psi_k\rangle \Psi_k(z)$. Analogously, the orthogonal projection operator of an arbitrary function $f \in \mathrm{H}^2(\mathbb{C}_-)$ on the subspace $\mathcal{R}_{\overline{N}}$ is $P_{\overline{N}} f(z) = \sum_{k=-N}^{-1}\langle f, \Psi_k\rangle \Psi_k(z)$.

Theorem 24. *[15] For any $f \in \mathrm{H}^2(\mathbb{C}_+)$ the projection operator $P_N f$ is an interpolation operator on the set $\{\lambda_k, \ 1 \le k \le N\}$, i.e. $P_N f(\lambda_k) = f(\lambda_k)$, $(1 \le k \le N)$. For any $f \in \mathrm{H}^2(\mathbb{C}_-)$ the projection operator $P_{\overline{N}} f$ is an interpolation operator on the set $\{\overline{\lambda_k}, \ 1 \le k \le N\}$, i.e. $P_{\overline{N}} f(\overline{\lambda_k}) = f(\overline{\lambda_k})$, $(1 \le k \le N)$.*

Theorem 25. *[15] Let suppose that condition (6.2) is satisfied. Then for any $f \in \mathrm{H}^2(\mathbb{C}_+)$ and any $z \in \mathbb{C}_+$ we have $P_N f(z) \to f(z)$, and for any $f \in \mathrm{H}^2(\mathbb{C}_-)$ and any $z \in \mathbb{C}_-$ we have $P_{\overline{N}} f(z) \to f(z)$ as $N \to \infty$.*

We are also interested in to know the behavior of P_N and $P_{\overline{N}}$ on the real line.

Theorem 26. *[15] If $f \in H^2(\mathbb{C}_+)$ has a partial fraction decomposition*

$$f(z) = \sum_{\ell=1}^{m} \frac{c_\ell}{z-\overline{\gamma_\ell}}, \quad \gamma_\ell \in \mathbb{C}_+, \tag{6.3}$$

then $|f(t) - P_N f(t)| \to 0$ uniformly on $\mathbb{R}$. Moreover $\lim\limits_{N\to\infty} \max_{t\in\mathbb{R}}(1+t^2)|f(t) - P_N f(t)|^2 \to 0$.

Analogously, if $f \in H^2(\mathbb{C}_-)$ has a partial fraction decomposition

$$f(z) = \sum_{\ell=1}^{m} \frac{c_\ell}{z-\gamma_\ell}, \quad \gamma_\ell \in \mathbb{C}_+, \tag{6.4}$$

then $|f(t) - P_{\overline{N}} f(t)| \to 0$ uniformly on $\mathbb{R}$ and $\lim\limits_{N\to\infty} \max_{t\in\mathbb{R}}(1+t^2)|f(t) - P_{\overline{N}} f(t)|^2 \to 0$.

The equation $\frac{z-a_1}{1-\overline{a_1}z} \cdot \frac{z-a_2}{1-\overline{a_2}z} \cdot \ldots \cdot \frac{z-a_N}{1-\overline{a_N}z} = 1$ has N different solutions and they can be written as $z_k := e^{i\tau_k}$, és $\tau_k := \theta_N^{-1}(2\pi(k-1)/N)$, $(k = 1,2,\ldots,N)$, where θ_N^{-1} is the inverse function of function $\theta_N(t) := \frac{1}{N}(\beta_{a_1}(t) + \ldots + \beta_{a_N}(t))$, $(t \in \mathbb{R})$.

Let us consider the set of (not uniformly distributed) nodes on the unit circle defined by

$$\mathbb{T}_N := \mathbb{T}_N^a := \{w_k = e^{i\tau_k} : \tau_k = \theta_N^{-1}(2\pi(k-1)/N), k = 1, 2, \ldots, N\}, \ (N = 1, 2, \ldots).$$

or the Malmquist-Takenaka system of the upper and lower half plane let denote $t_k = \tan\frac{\tau_k}{2}$, where τ_k, $(k = 1, \ldots, N)$ and $z_k = e^{i\tau_k} = \frac{i-t_k}{i+t_k}$, $(k = 1, \ldots, N)$. Let us introduce the following set of nodes on the real line $\mathbb{R}_N = \{t_k : \ k = 1, \ldots, N\}$.

Suppose that every node is finite. Let define the following weight function:

$$\frac{1}{\widetilde{\rho}_N(t)} := K_N(t,t) = \sum_{k=1}^{N} \frac{\Im\lambda_k}{\pi|t - \overline{\lambda_k}|^2} \quad (t \in \mathbb{R}), \tag{6.5}$$

and the discrete scalar product $\langle F, G\rangle_N = \sum_{t\in\mathbb{R}_N} F(t)\overline{G(t)}\widetilde{\rho}_N(t)$.

Theorem 27. *[15] The finite collection of $\{\Psi_n\}_{n=1}^N$ forms a discrete orthonormal system with respect to the scalar product defined above namely $\langle \Psi_n, \Psi_m\rangle_N = \delta_{mn}$ $(1 \le m, n \le N)$.*

Similarly, the finite collection of $\{\Psi_n\}_{n=-N}^{-1}$ forms a discrete orthonormal system with respect to the scalar product $\langle ., .\rangle_N$, i.e. $\langle \Psi_n, \Psi_m\rangle_N = \delta_{mn}$ $(-N \le m, n \le -1)$.

Let consider $\mathbb{A}(\mathbb{C}_+)$, the upper half plane algebra of analytic functions, consisting those functions which are analytic on $\mathbb{C}_+$, continuous on $\overline{\mathbb{C}_+}$, and with boundary function satisfying $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0$.

Considering as nodes $\mathbb{R}_N$ let consider the following interpolation operator:

$$\mathcal{L}_N f := \sum_{t\in\mathbb{R}_N} \frac{K_N(.,t)}{K_N(t,t)} f(t), \quad \text{ahol} \quad f \in \mathbb{A}(\mathbb{C}_+).$$

Analogously for the lower half plane algebra of analytic functions $\mathbb{A}(\mathbb{C}_-)$ let consider the following interpolation operator:

$$\mathcal{L}_{\overline{N}} f := \sum_{t\in\mathbb{R}_N} \frac{K_{\overline{N}}(.,t)}{K_{\overline{N}}(t,t)} f(t),$$

where $f \in \mathbb{A}(\mathbb{C}_-)$.

$\mathcal{L}_N f$ and $\mathcal{L}_{\overline{N}} f$ interpolate f at the points of $\mathbb{R}_N$.

These operators are exact on $\mathcal{R}_N$ and $\mathcal{R}_{\overline{N}}$ respectively, i.e.,

$$\mathcal{L}_N f = P_N f = f, \ f \in \mathcal{R}_N, \quad \mathcal{L}_{\overline{N}} f = P_{\overline{N}} f = f, \ f \in \mathcal{R}_{\overline{N}}. \tag{6.6}$$

As a consequence of the previous property we can propose a new exact interpolation scheme for those functions which belong to $\mathcal{R}_{N,\overline{N}}$. Let $f \in \mathcal{R}_{N,\overline{N}}$, then $f = f_1 + f_2$, where $f_1 \in \mathcal{R}_N$ and $f_2 \in \mathcal{R}_{\overline{N}}$ and let define $L_N f = \mathcal{L}_N f_1 + \mathcal{L}_{\overline{N}} f_2$. Then for every $f \in \mathcal{R}_{N,\overline{N}}$

$$L_N f = \mathcal{L}_N f_1 + \mathcal{L}_{\overline{N}} f_2 = f_1 + f_2 = f.$$

If we choose $\lambda_1 = i$, then the Runge's test function belongs to $\mathcal{R}_{N,\overline{N}}$. Indeed $f(z) = \frac{1}{z^2+1} = \frac{1}{2i(z-i)} - \frac{1}{2i(z+i)}$.

Taking $f_1 = \frac{-1}{2i(z+i)} \in \mathcal{R}_N$ and $f_2 = \frac{1}{2i(z-i)} \in \mathcal{R}_{\overline{N}}$ we obtain the following exact interpolation for the Runge's function: $L_N f = \mathcal{L}_N f_1 + \mathcal{L}_{\overline{N}} f_2 = f_1 + f_2 = f$.

Theorem 28. *[15] Let $\lambda_1 = i$, $\lambda_k \in \mathbb{C}_+$ such that $\sum_{k=1}^{\infty} \frac{\Im\lambda_k}{1+|\lambda_k|^2} = \infty$. If $f \in \mathbb{A}(\mathbb{C}_+)$ is uniformly continuous on $\overline{\mathbb{C}_+}$ such that $\lim\limits_{N\to\infty} \max_{t\in\mathbb{R}}(1+t^2)|f(t) - P_{\overline{N}} f(t)|^2 = 0$, then the interpolation operator $\mathcal{L}_{\overline{N}} f := \sum_{t\in\mathbb{R}_N} \frac{K_{\overline{N}}(.,t)}{K_{\overline{N}}(t,t)} f(t)$ converges to f in norm, i.e., $\lim\limits_{N\to\infty} \|f - \mathcal{L}_{\overline{N}} f\|_2 = 0$.*

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